University of Anbar

Engineering College

Department of Mechanical Engineering

ME 3301 - Engineering Analysis (3-3-1-0)

Third Stage

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Reference: Dennis G. Zill, Warren S. Wright, (2012)-Advanced Engineering Mathematics

Course Topics

1. Introduction to Differential Equations
2. Modeling with Higher Order Linear Differential Equations.
3. Systems of Differential Equations.
4. Applications of Ordinary Differential Equations.
5. Fourier series
6. Partial Differential Equations.
7. Functions of complex variables

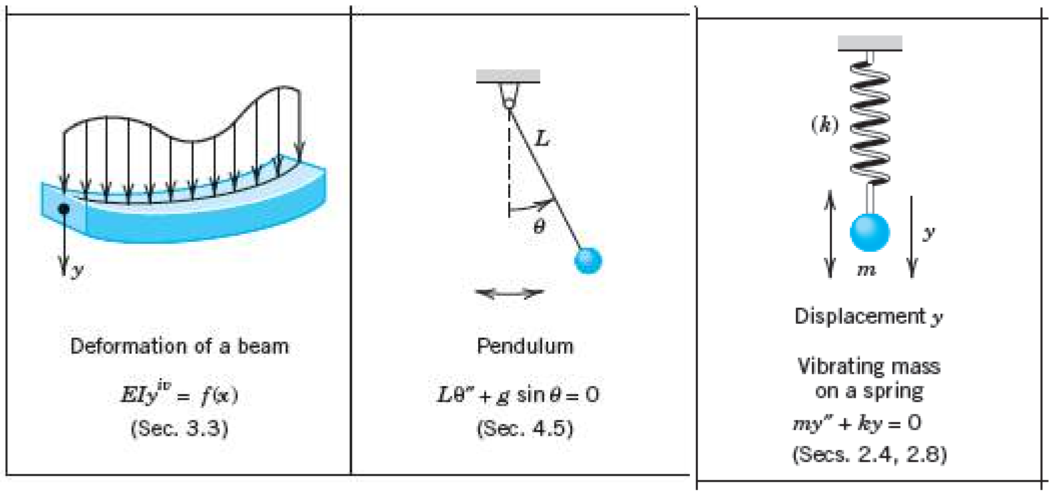
Course Learning Outcomes:

By the end of successful completion of this course, the student will be able to:

1. Think logically and mathematically for solving practical problems such as mechanical vibrations, fluid flow problems, heat transfer problems.
2. Practice modeling and be able to translate engineering and physical situations into a mathematical model
3. To gain experience and further mastery of complete problem, solving fluency based on Fourier Series and Partial Differential Equations.
4. Use proper assumptions to describe the complex behaviour of practical problems and able to read and interpret problem objectives.
5. Realize modelling with partial differential equations and Fourier analysis for solving various practical applications

Chapter 1: Modeling with Higher Order Linear Differential Equations.

To solve an engineering problem, we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical model of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called mathematical modeling



1 Separable ODEs. Modeling

Many practically useful ODEs can be reduced to the form

(1) $$g(y)\, y' = f(x)$$

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

(2) $$\int g(y)\, y' dx = \int f(x)\, dx + c.$$

On the left we can switch to y as the variable of integration. By calculus, $y' dx = dy$, so that

(3) $$\int g(y)\, dy = \int f(x)\, dx + c.$$

EXAMPLE 1 Separable ODE

The ODE $y' = 1 + y^2$ is separable because it can be written

$$\frac{dy}{1+y^2} = dx. \qquad \text{By integration,} \qquad \arctan y = x + c \qquad \text{or} \qquad y = \tan(x + c).$$

Example Heating an Office Building (Newton's Law of Cooling)

Suppose that in winter the daytime temperature in a certain office building is maintained at 70°F. The heating is shut off at 10 P.M. and turned on again at 6 A.M. On a certain day the temperature inside the building at 2 A.M. was found to be 65°F. The outside temperature was 50°F at 10 P.M. and had dropped to 40°F by 6 A.M. What was the temperature inside the building when the heat was turned on at 6 A.M.?

Solution. ***Step 1. Setting up a model.*** Let $T(t)$ be the temperature inside the building and T_A the outside temperature (assumed to be constant in Newton's law). Then by Newton's law,

$$\frac{dT}{dt} = k(T - T_A). \tag{6}$$

Step 2. General solution. We cannot solve (6) because we do not know T_A, just that it varied between 50°F and 40°F, so we follow the ***Golden Rule:*** *If you cannot solve your problem, try to solve a simpler one.* We solve (6) with the unknown function T_A replaced with the average of the two known values, or 45°F. For physical reasons we may expect that this will give us a reasonable approximate value of T in the building at 6 A.M.

For constant $T_A = 45$ (or any other *constant* value) the ODE (6) is separable. Separation, integration, and taking exponents gives the general solution

$$\frac{dT}{T-45} = k\,dt, \qquad \ln|T - 45| = kt + c^*, \qquad T(t) = 45 + ce^{kt} \qquad (c = e^{c^*}).$$

Step 3. Particular solution. We choose 10 P.M. to be $t = 0$. Then the given initial condition is $T(0) = 70$ and yields a particular solution, call it T_p. By substitution,

$$T(0) = 45 + ce^0 = 70, \qquad c = 70 - 45 = 25, \qquad T_p(t) = 45 + 25e^{kt}.$$

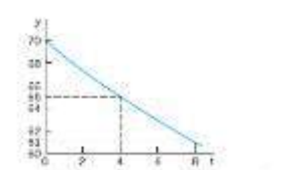
Step 4. Determination of k. We use $T(4) = 65$, where $t = 4$ is 2 A.M. Solving algebraically for k and inserting k into $T_p(t)$ gives (Fig. 12)

$$T_p(4) = 45 + 25e^{4k} = 65, \qquad e^{4k} = 0.8, \qquad k = \tfrac{1}{4}\ln 0.8 = -0.056, \qquad T_p(t) = 45 + 25e^{-0.056t}.$$

Step 5. Answer and interpretation. 6 A.M. is $t = 8$ (namely, 8 hours after 10 P.M.), and

$$T_p(8) = 45 + 25e^{-0.056 \cdot 8} = 61[°\text{F}].$$

Hence the temperature in the building dropped 9°F, a result that looks reasonable.



Particular solution (temperature) in Example 6

Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law)

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 13). You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

Physical information. Under the influence of gravity the outflowing water has velocity

$$v(t) = 0.600\sqrt{2gh(t)} \qquad \textbf{(Torricelli's law[4])}, \tag{7}$$

where $h(t)$ is the height of the water above the hole at time t, and $g = 980\ \text{cm/sec}^2 = 32.17\ \text{ft/sec}^2$ is the acceleration of gravity at the surface of the earth.

Solution. ***Step 1. Setting up the model.*** To get an equation, we relate the decrease in water level $h(t)$ to the outflow. The volume ΔV of the outflow during a short time Δt is

$$\Delta V = Av\ \Delta t \qquad (A = \text{Area of hole}).$$

ΔV must equal the change ΔV^* of the volume of the water in the tank. Now

$$\Delta V^* = -B\ \Delta h \qquad (B = \text{Cross-sectional area of tank})$$

where Δh (> 0) is the decrease of the height $h(t)$ of the water. The minus sign appears because the volume of the water in the tank decreases. Equating ΔV and ΔV^* gives

$$-B\ \Delta h = Av\ \Delta t.$$

We now express v according to Torricelli's law and then let Δt (the length of the time interval considered) approach 0—this is a ***standard way*** of obtaining an ODE as a model. That is, we have

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}v = -\frac{A}{B}0.600\sqrt{2gh(t)}$$

and by letting $\Delta t \to 0$ we obtain the ODE

$$\frac{dh}{dt} = -26.56\frac{A}{B}\sqrt{h},$$

where $26.56 = 0.600\sqrt{2 \cdot 980}$. This is our model, a first-order ODE.

Step 2. General solution. Our ODE is separable. A/B is constant. Separation and integration gives

$$\frac{dh}{\sqrt{h}} = -26.56\frac{A}{B}dt \qquad \text{and} \qquad 2\sqrt{h} = c^* - 26.56\frac{A}{B}t.$$

Dividing by 2 and squaring gives $h = (c - 13.28At/B)^2$. Inserting $13.28A/B = 13.28 \cdot 0.5^2\pi/100^2\pi = 0.000\,332$ yields the general solution

$$h(t) = (c - 0.000\,332t)^2.$$

Step 3. Particular solution. The initial height (the initial condition) is $h(0) = 225$ cm. Substitution of $t = 0$ and $h = 225$ gives from the general solution $c^2 = 225$, $c = 15.00$ and thus the particular solution (Fig. 13)

$$h_p(t) = (15.00 - 0.000332t)^2.$$

Step 4. Tank empty. $h_p(t) = 0$ if $t = 15.00/0.000332 = 45{,}181$ [sec] $= 12.6$ [hours].

Here you see distinctly the ***importance of the choice of units***—we have been working with the cgs system, in which time is measured in seconds! We used $g = 980\ \text{cm/sec}^2$.

Step 5. Checking. Check the result.

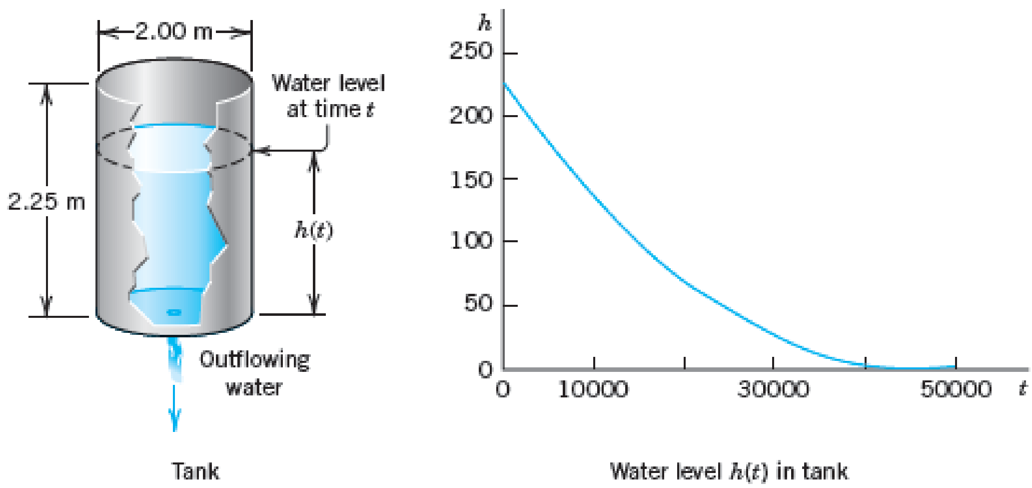


Fig. 13. Example 7. Outflow from a cylindrical tank ("leaking tank"). Torricelli's law

2 Exact ODEs.

A first-order ODE $M(x, y) + N(x, y)y' = 0$, written as (use $dy = y'dx$ as in Sec. 1.3)

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact**, that is, this form is the differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{2}$$

of some function $u(x, y)$. Then (1) can be written

$$du = 0.$$

By integration we immediately obtain the general solution of (1) in the form

$$u(x, y) = c. \tag{3}$$

This is called an implicit solution

Comparing (1) and (2), we see that (1) is an exact differential equation if there is some function $u(x, y)$ such that

$$\text{(a)}\quad \frac{\partial u}{\partial x} = M, \qquad \text{(b)}\quad \frac{\partial u}{\partial y} = N. \tag{4}$$

The condition to be an exact differential equation is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If (1) is exact, the function $u(x, y)$ can be found by inspection or in the following systematic way. From (4a) we have by integration with respect to x

$$u = \int M\,dx + k(y); \tag{6}$$

in this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\partial u/\partial y$ from (6), use (4b) to get dk/dy, and integrate dk/dy to get k. (See Example 1, below.)

Formula (6) was obtained from (4a). Instead of (4a) we may equally well use (4b). Then, instead of (6), we first have by integration with respect to y

$$u = \int N\,dy + l(x). \tag{6*}$$

To determine $l(x)$, we derive $\partial u/\partial x$ from (6*), use (4a) to get dl/dx, and integrate. We illustrate all this by the following typical examples.

EXAMPLE 1 An Exact ODE

Solve

$$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0. \tag{7}$$

Solution. ***Step 1. Test for exactness.*** Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

$$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y). \tag{8}$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

Step 3. Checking an implicit solution. We can check by differentiating the implicit solution $u(x, y) = c$ implicitly and see whether this leads to the given ODE (7):

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \cos(x + y)\, dx + (\cos(x + y) + 3y^2 + 2y)\, dy = 0. \tag{9}$$

3 Linear ODEs.

A first-order ODE is said to be linear if it can be brought into the form

$$y' + p(x)y = r(x), \tag{1}$$

Homogeneous Linear ODE. We want to solve (1) in some interval $a < x < b$, call it J, and we begin with the simpler special case that $r(x)$ is zero for all x in J. (This is sometimes written $r(x) \equiv 0$.) Then the ODE (1) becomes

$$y' + p(x)y = 0 \tag{2}$$

and is called **homogeneous**. By separating variables and integrating we then obtain

$$\frac{dy}{y} = -p(x)\,dx, \qquad \text{thus} \qquad \ln|y| = -\int p(x)\,dx + c^*.$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE (2),

$$y(x) = ce^{-\int p(x)dx} \qquad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0); \tag{3}$$

Nonhomogeneous Linear ODE. We now solve (1) in the case that $r(x)$ in (1) is not everywhere zero in the interval J considered. Then the ODE (1) is called **nonhomogeneous**.

The desired solution formula

$$y(x) = e^{-h}\left(\int e^{h} r\, dx + c\right), \qquad h = \int p(x)\, dx. \tag{4}$$

Example 1:

Solve the initial value problem

$$y' + y \tan x = \sin 2x, \qquad y(0) = 1.$$

Solution. Here $p = \tan x$, $r = \sin 2x = 2 \sin x \cos x$, and

$$h = \int p\, dx = \int \tan x\, dx = \ln |\sec x|.$$

From this we see that in (4),

$$e^h = \sec x, \qquad e^{-h} = \cos x, \qquad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x,$$

and the general solution of our equation is

$$y(x) = \cos x \left(2 \int \sin x\, dx + c\right) = c \cos x - 2 \cos^2 x.$$

From this and the initial condition, $1 = c \cdot 1 - 2 \cdot 1^2$; thus $c = 3$ and the solution of our initial value problem is $y = 3 \cos x - 2 \cos^2 x$. Here $3 \cos x$ is the response to the initial data, and $-2 \cos^2 x$ is the response to the input $\sin 2x$. ■

Example 2:

Electric Circuit

Model the ***RL*-circuit** in Fig. 19 and solve the resulting ODE for the current $I(t)$ A (amperes), where t is time. Assume that the circuit contains as an EMF $E(t)$ (electromotive force) a battery of $E = 48$ V (volts), which is constant, a ***resistor*** of $R = 11\ \Omega$ (ohms), and an ***inductor*** of $L = 0.1$ H (henrys), and that the current is initially zero.

Physical Laws. A current I in the circuit causes a **voltage drop** RI across the resistor (**Ohm's law**) and a voltage drop $LI' = L\, dI/dt$ across the conductor, and the sum of these two voltage drops equals the EMF (**Kirchhoff's Voltage Law, KVL**).

Solution. According to these laws the model of the RL-circuit is $LI' + RI = E(t)$, in standard form

$$I' + \frac{R}{L} I = \frac{E(t)}{L}. \tag{6}$$

We can solve this linear ODE by (4) with $x = t$, $y = I$, $p = R/L$, $h = (R/L)t$, obtaining the general solution

$$I = e^{-(R/L)t} \left(\int e^{(R/L)t} \frac{E(t)}{L} dt + c \right).$$

By integration,

$$I = e^{-(R/L)t} \left(\frac{E}{L} \frac{e^{(R/L)t}}{R/L} + c \right) = \frac{E}{R} + c e^{-(R/L)t}. \tag{7}$$

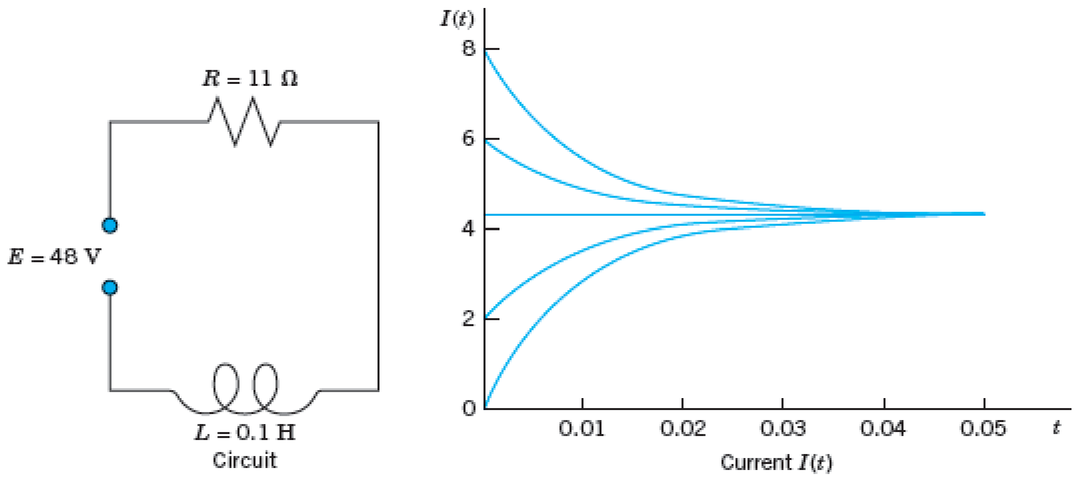
In our case, $R/L = 11/0.1 = 110$ and $E(t) = 48/0.1 = 480 = \text{const}$; thus,

$$I = \tfrac{48}{11} + |ce^{-110t}.$$

In modeling, one often gets better insight into the nature of a solution (and smaller roundoff errors) by inserting given numeric data only near the end. Here, the general solution (7) shows that the current approaches the limit $E/R = 48/11$ faster the larger R/L is, in our case, $R/L = 11/0.1 = 110$, and the approach is very fast, from below if $I(0) < 48/11$ or from above if $I(0) > 48/11$. If $I(0) = 48/11$, the solution is constant (48/11 A). See Fig. 19.

The initial value $I(0) = 0$ gives $I(0) = E/R + c = 0$, $c = -E/R$ and the particular solution

(8) $$I = \frac{E}{R}(1 - e^{-(R/L)t}), \qquad \text{thus} \qquad I = \frac{48}{11}(1 - e^{-110t}).$$ ■



4 Homogeneous linear equations with constant coefficients;

For a linear differential **equation**, an nth-order initial-value problem is

$$\text{Solve:} \quad a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

$$\text{Subject to:} \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}.$$

with *g(x)* not identically zero, is said to be nonhomogeneous and it will be homogeneous, when *g(x)=0*

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

 Auxiliary Equation We begin by considering the special case of a second-order equation

$$ay'' + by' + cy = 0. \tag{2}$$

If we try a solution of the form $y = e^{mx}$, then after substituting $y' = me^{mx}$ and $y'' = m^2e^{mx}$ equation (2) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

Since e^{mx} is never zero for real values of x, it is apparent that the only way that this exponential function can satisfy the differential equation (2) is to choose m as a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (1) corresponding to the three cases:

- m_1 and m_2 are real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 are real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 are conjugate complex numbers ($b^2 - 4ac < 0$).

Case I*: *Distinct Real Roots Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions, $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$, respectively. We see that these functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1e^{m_1x} + c_2e^{m_2x}. \tag{4}$$

Case II*: *Repeated Real Roots When $m_1 = m_2$ we necessarily obtain only one exponential solution, $y_1 = e^{m_1x}$. From the quadratic formula we find that $m_1 = -b/2a$ since the only way to have $m_1 = m_2$ is to have $b^2 - 4ac = 0$. It follows from the discussion in Section 3.2 that a second solution of the equation is

$$y_2 = e^{m_1x}\int \frac{e^{2m_1x}}{e^{2m_1x}}\,dx = e^{m_1x}\int dx = xe^{m_1x}. \tag{5}$$

In (5) we have used the fact that $-b/a = 2m_1$. The general solution is then

$$y = c_1e^{m_1x} + c_2xe^{m_1x}. \tag{6}$$

Case III*: *Conjugate Complex Roots If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real and $i^2 = -1$. Formally, there is no difference between this case and Case I, hence

$$y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}.$$

However, in practice we prefer to work with real functions instead of complex exponentials. To this end we use **Euler's formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is any real number.* It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x, \tag{7}$$

where we have used $\cos(-\beta x) = \cos\beta x$ and $\sin(-\beta x) = -\sin\beta x$. Note that by first adding and then subtracting the two equations in (7), we obtain, respectively,

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \quad \text{and} \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

Since $y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}$ is a solution of (2) for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1$, $C_2 = -1$ give, in turn, two solutions:

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}.$$

But $$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$

and $$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x.$$

Hence from Corollary (a) of Theorem 3.1.2 the last two results show that $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are *real* solutions of (2). Moreover, these solutions form a fundamental set on $(-\infty, \infty)$. Consequently, the general solution is

$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x). \tag{8}$$

EXAMPLE 1 **Second-Order DEs**

Solve the following differential equations.

(a) $2y'' - 5y' - 3y \quad 0$ **(b)** $y'' - 10y' + 25y \quad 0$ **(c)** $y'' + 4y' + 7y \quad 0$

SOLUTION We give the auxiliary equations, the roots, and the corresponding general solutions.

(a) $2\ \ ^2 - 5\ \ - 3 \quad (2\ \ + 1)(\ \ - 3),\ \ _1 \quad -\frac{1}{2},\ \ _2 \quad 3$. From (4),

$$y \quad c_1e^{-x/2} + c_2e^{3x}.$$

(b) $\ ^2 - 10\ \ + 25 \quad (\ \ - 5)^2,\ \ _1 \quad\ _2 \quad 5$. From (6),

$$y \quad c_1e^{5x} + c_2xe^{5x}.$$

(c) $\ ^2 + 4\ \ + 7 \quad 0,\ \ _1 \quad -2 + \sqrt{3}i,\ \ _2 \quad -2 - \sqrt{3}i$. From (8) with $\alpha \quad -2$, $\beta \quad \sqrt{3}$, we have

$$y \quad e^{-2x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x).$$ ≡

Solve the initial value problem

$$y'' + y' + 0.25y = 0, \qquad y(0) = 3.0, \quad y'(0) = -3.5.$$

Solution. The characteristic equation is $\lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0$. It has the double root $\lambda = -0.5$. This gives the general solution

$$y = (c_1 + c_2 x)e^{-0.5x}.$$

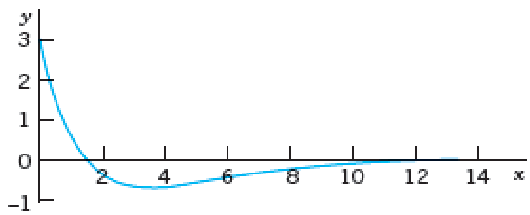
We need its derivative

$$y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2 x)e^{-0.5x}.$$

From this and the initial conditions we obtain

$$y(0) = c_1 = 3.0, \qquad y'(0) = c_2 - 0.5c_1 = 3.5; \qquad \text{hence} \qquad c_2 = -2.$$

The particular solution of the initial value problem is $y = (3 - 2x)e^{-0.5x}$. See Fig. 31. ■



Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Solution. ***Step 1. General solution.*** The characteristic equation is $\lambda^2 + 0.4\lambda + 9.04 = 0$. It has the roots $-0.2 \pm 3i$. Hence $\omega = 3$, and a general solution (9) is

$$y = e^{-0.2x}(A \cos 3x + B \sin 3x).$$

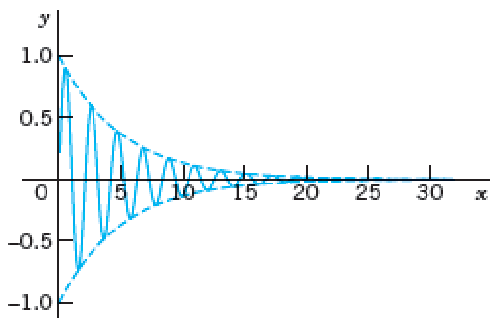
Step 2. Particular solution. The first initial condition gives $y(0) = A = 0$. The remaining expression is $y = Be^{-0.2x} \sin 3x$. We need the derivative (chain rule!)

$$y' = B(-0.2e^{-0.2x} \sin 3x + 3e^{-0.2x} \cos 3x).$$

From this and the second initial condition we obtain $y'(0) = 3B = 3$. Hence $B = 1$. Our solution is

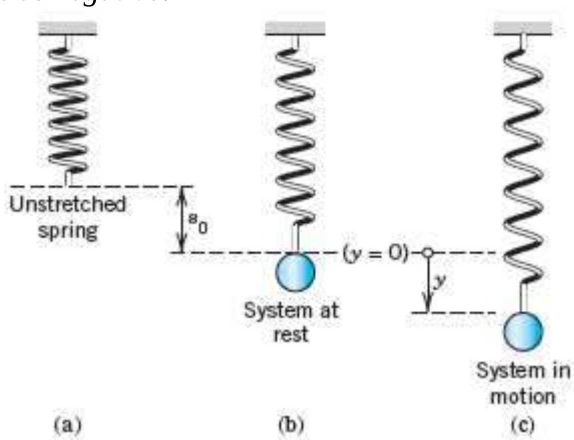
$$y = e^{-0.2x} \sin 3x.$$

Figure 32 shows y and the curves of $e^{-0.2x}$ and $-e^{-0.2x}$ (dashed), between which the curve of y oscillates. Such "damped vibrations" (with $x = t$ being time) have important mechanical and electrical applications, as we shall soon see (in Sec. 2.4).



5 Modeling of Free Oscillations of a Mass–Spring System

We take an ordinary coil spring that resists extension as well as compression. We suspend it vertically from a fixed support and attach a body at its lower end, for instance, an iron ball, as shown in Fig. 33. We let y=0 denote the position of the ball when the system is at rest (Fig. 33b). Furthermore, we choose ***the downward direction as positive***, thus regarding downward forces as *positive* and upward forces as *negative*.



We now let the ball move, as follows. We pull it down by an amount $y > 0$ (Fig. 33c). This causes a spring force

$$F_1 = -ky \qquad (1) \qquad \textbf{(Hooke's law}^2\textbf{)}$$

The motion of our mass–spring system is determined by **Newton's second law**

$$\text{Mass} \times \text{Acceleration} = my'' = \text{Force} \qquad (2)$$

where $y'' = d^2y/dt^2$ and "Force" is the resultant of all the forces acting on the ball.

ODE of the Undamped System

Every system has damping. Otherwise it would keep moving forever. But if the damping is small and the motion of the system is considered over a relatively short time, we may disregard damping. Then Newton's law with $F = -F_1$ gives the model $my'' = -F_1 = -ky$; thus

$$my'' + ky = 0. \qquad (3)$$

This is a homogeneous linear ODE with constant coefficients. A general solution is obtained as in Sec. 2.2, namely (see Example 6 in Sec. 2.2)

$$y(t) = A \cos \omega_0 t + B \sin \omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}. \qquad (4)$$

This motion is called a **harmonic oscillation** (Fig. 34). Its *frequency* is $f = \omega_0/2\pi$ Hertz[3] (= cycles/sec) because cos and sin in (4) have the period $2\pi/\omega_0$. The frequency f is called the **natural frequency** of the system. (We write ω_0 to reserve ω for Sec. 2.8.)

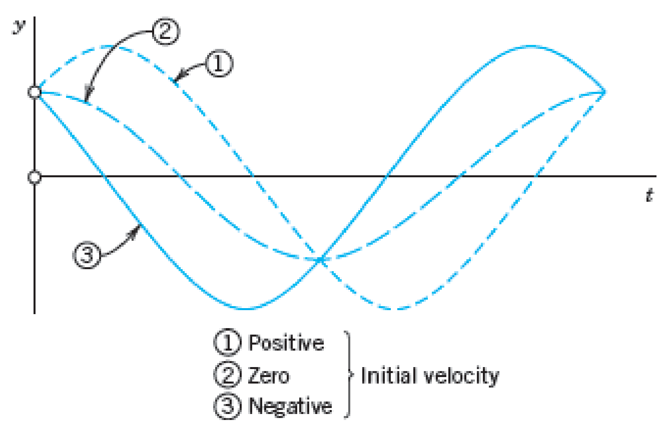


Fig. 34. Typical harmonic oscillations (4) and (4*) with the same $y(0) = A$ and different initial velocities $y'(0) = \omega_0 B$, positive ①, zero ②, negative ③

An alternative representation of (4), which shows the physical characteristics of amplitude and phase shift of (4), is

(4*) $$y(t) = C\cos(\omega_0 t - \delta)$$

with $C = \sqrt{A^2 + B^2}$ and phase angle δ, where $\tan\delta = B/A$. This follows from the addition formula (6) in App. 3.1.

Example

Harmonic Oscillation of an Undamped Mass–Spring System

If a mass–spring system with an iron ball of weight $W = 98$ nt (about 22 lb) can be regarded as undamped, and the spring is such that the ball stretches it 1.09 m (about 43 in.), how many cycles per minute will the system execute? What will its motion be if we pull the ball down from rest by 16 cm (about 6 in.) and let it start with zero initial velocity?

Solution. Hooke's law (1) with W as the force and 1.09 meter as the stretch gives $W = 1.09k$; thus $k = W/1.09 = 98/1.09 = 90$ [kg/sec^2] $= 90$ [nt/meter]. The mass is $m = W/g = 98/9.8 = 10$ [kg]. This gives the frequency $\omega_0/(2\pi) = \sqrt{k/m}/(2\pi) = 3/(2\pi) = 0.48$ [Hz] $= 29$ [cycles/min].

From (4) and the initial conditions, $y(0) = A = 0.16$ [meter] and $y'(0) = \omega_0 B = 0$. Hence the motion is

$$y(t) = 0.16\cos 3t \text{ [meter]} \quad \text{or} \quad 0.52\cos 3t \text{ [ft]} \qquad \text{(Fig. 35).}$$

ODE of the Damped System

To our model $my'' = -ky$ we now add a damping force

$$F_2 = -cy',$$

obtaining $my'' = -ky - cy'$; thus the ODE of the damped mass–spring system is

(5) $$my'' + cy' + ky = 0. \qquad \text{(Fig. 36)}$$

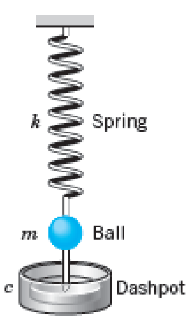


Fig. 36. Damped system

Physically this can be done by connecting the ball to a dashpot; see Fig. 36. We assume this damping force to be proportional to the velocity $y' = dy/dt$. This is generally a good approximation for small velocities.

The constant c is called the *damping constant*. Let us show that c is positive. Indeed, the damping force $F_2 = -cy'$ acts *against* the motion; hence for a downward motion we have $y' > 0$ which for positive c makes F negative (an upward force), as it should be. Similarly, for an upward motion we have $y' < 0$ which, for $c > 0$ makes F_2 positive (a downward force).

The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by m)

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$(6) \quad \lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta, \quad \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}.$$

It is now interesting that depending on the amount of damping present—whether a lot of damping, a medium amount of damping or little damping—three types of motions occur, respectively:

Case I.	$c^2 > 4mk$.	*Distinct real roots* λ_1, λ_2.	**(Overdamping)**
Case II.	$c^2 = 4mk$.	*A real double root.*	**(Critical damping)**
Case III.	$c^2 < 4mk$.	*Complex conjugate roots.*	**(Underdamping)**

Case I. Overdamping

If the damping constant c is so large that $c^2 > 4mk$, then λ_1 and λ_2 are distinct real roots. In this case the corresponding general solution of (5) is

$$(7) \quad y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}.$$

We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0, \beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically speaking, after a sufficiently long time the mass will be at rest at the *static equilibrium position* ($y = 0$). Figure 37 shows (7) for some typical initial conditions.

Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}. \tag{8}$$

Case III. Underdamping

This is the most interesting case. It occurs if the damping constant c is so small that $c^2 < 4mk$. Then β in (6) is no longer real but pure imaginary, say,

$$\beta = i\omega^* \qquad \text{where} \qquad \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \qquad (>0). \tag{9}$$

(We now write ω^* to reserve ω for driving and electromotive forces in Secs. 2.8 and 2.9.) The roots of the characteristic equation are now complex conjugates,

$$\lambda_1 = -\alpha + i\omega^*, \qquad \lambda_2 = -\alpha - i\omega^*$$

with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

$$y(t) = e^{-\alpha t}(A \cos \omega^* t + B \sin \omega^* t) = Ce^{-\alpha t} \cos (\omega^* t - \delta) \tag{10}$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*).

This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 39, touching them when $\omega^* t - \delta$ is an integer multiple of π because these are the points at which $\cos (\omega^* t - \delta)$ equals 1 or -1.

The frequency is $\omega^*/(2\pi)$ Hz (hertz, cycles/sec). From (9) we see that the smaller c (>0) is, the larger is ω^* and the more rapid the oscillations become. If c approaches 0, then ω^* approaches $\omega_0 = \sqrt{k/m}$, giving the harmonic oscillation (4), whose frequency $\omega_0/(2\pi)$ is the natural frequency of the system.

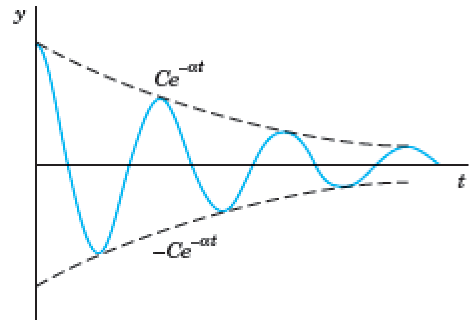


Fig. 39. Damped oscillation in Case III [see (10)]

Example

How does the motion in Example 1 change if we change the damping constant c from one to another of the following three values, with $y(0) = 0.16$ and $y'(0) = 0$ as before?

$$\text{(I) } c = 100 \text{ kg/sec}, \qquad \text{(II) } c = 60 \text{ kg/sec}, \qquad \text{(III) } c = 10 \text{ kg/sec}.$$

Solution. It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).

(I) With $m = 10$ and $k = 90$, as in Example 1, the model is the initial value problem

$$10y'' + 100y' + 90y = 0, \qquad y(0) = 0.16 \text{ [meter]}, \qquad y'(0) = 0.$$

The characteristic equation is $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$. It has the roots -9 and -1. This gives the general solution

$$y = c_1 e^{-9t} + c_2 e^{-t}. \qquad \text{We also need} \qquad y' = -9c_1 e^{-9t} - c_2 e^{-t}.$$

The initial conditions give $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$

It approaches 0 as $t \to \infty$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.

(II) The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form $10\lambda^2 + 60\lambda + 90 = 10(\lambda + 3)^2 = 0$. It has the double root -3. Hence the corresponding general solution is

$$y = (c_1 + c_2 t)e^{-3t}. \qquad \text{We also need} \qquad y' = (c_2 - 3c_1 - 3c_2 t)e^{-3t}.$$

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$. Hence in the critical case the solution is

$$y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.

(III) The model now is $10y'' + 10y' + 90y = 0$. Since $c = 10$ is smaller than the critical c, we shall get oscillations. The characteristic equation is $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots [see (4) in Sec. 2.2 with $a = 1$ and $b = 9$]

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$

This gives the general solution

$$y = e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative

$$y' = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t).$$

Hence $y'(0) = -0.5A + 2.96B = 0, \quad B = 0.5A/2.96 = 0.027$. This gives the solution

$$y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) = 0.162e^{-0.5t} \cos (2.96t - 0.17).$$

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero. See Fig. 40. ■

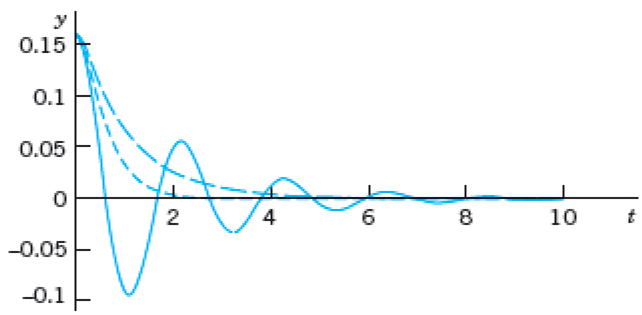


Fig. 40. The three solutions in Example 2

6 Modeling: Forced Oscillations. Resonance

$$my'' + cy' + ky = 0.$$

We now extend our model by including an additional force, that is, the external force $r(t)$, on the right. Then we have

$$my'' + cy' + ky = r(t). \quad (2^*)$$

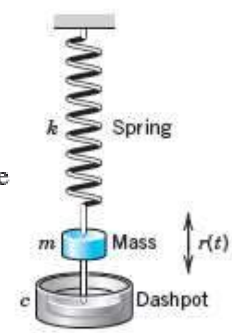


Fig. 53. Mass on a spring

Mechanically this means that at each instant t the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a **forced motion** with **forcing function** $r(t)$, which is also known as **input** or **driving force**, and the solution $y(t)$ to be obtained is called the **output** or the **response** *of the system to the driving force.*

Of special interest are periodic external forces, and we shall consider a driving force of the form

$$r(t) = F_0 \cos \omega t \qquad (F_0 > 0, \omega > 0).$$

Then we have the nonhomogeneous ODE

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{2}$$

Solving the Nonhomogeneous ODE (2)

From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution y_h of the homogeneous ODE (1) plus any solution y_p of (2). To find y_p, we use the method of undetermined coefficients (Sec. 2.7), starting from

$$y_p(t) = a \cos \omega t + b \sin \omega t. \tag{3}$$

By differentiating this function (chain rule!) we obtain

$$y_p' = -\omega a \sin \omega t + \omega b \cos \omega t,$$

$$y_p'' = -\omega^2 a \cos \omega t - \omega^2 b \sin \omega t.$$

Substituting y_p, y_p', and y_p'' into (2) and collecting the cosine and the sine terms, we get

$$[(k - m\omega^2)a + \omega cb] \cos \omega t + [-\omega ca + (k - m\omega^2)b] \sin \omega t = F_0 \cos \omega t.$$

The cosine terms on both sides must be equal, and the coefficient of the sine term on the left must be zero since there is no sine term on the right. This gives the two equations

$$\begin{aligned} (k - m\omega^2)a + \quad \omega cb \quad &= F_0 \\ -\omega ca \quad + (k - m\omega^2)b &= 0 \end{aligned} \tag{4}$$

for determining the unknown coefficients a and b. This is a linear system. We can solve it by elimination. To eliminate b, multiply the first equation by $k - m\omega^2$ and the second by $-\omega c$ and add the results, obtaining

$$(k - m\omega^2)^2 a + \omega^2 c^2 a = F_0(k - m\omega^2).$$

Similarly, to eliminate a, multiply (the first equation by ωc and the second by $k - m\omega^2$ and add to get

$$\omega^2 c^2 b + (k - m\omega^2)^2 b = F_0 \omega c.$$

If the factor $(k - m\omega^2)^2 + \omega^2 c^2$ is not zero, we can divide by this factor and solve for a and b,

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}.$$

If we set $\sqrt{k/m} = \omega_0$ (> 0) as in Sec. 2.4, then $k = m\omega_0^2$ and we obtain

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \qquad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}. \tag{5}$$

We thus obtain the general solution of the nonhomogeneous ODE (2) in the form

$$y(t) = y_h(t) + y_p(t). \tag{6}$$

Case 1. Undamped Forced Oscillations. Resonance

If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$. Hence (3) becomes (use $\omega_0{}^2 = k/m$)

$$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t. \tag{7}$$

Here we must assume that $\omega^2 \neq \omega_0{}^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4]. From (7) and from (4*) in Sec. 2.4 we have the general solution of the "undamped system"

$$y(t) = C \cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t. \tag{8}$$

Resonance. We discuss (7). We see that the maximum amplitude of y_p is (put $\cos \omega t = 1$)

$$a_0 = \frac{F_0}{k}\rho \qquad \text{where} \qquad \rho = \frac{1}{1-(\omega/\omega_0)^2}. \tag{9}$$

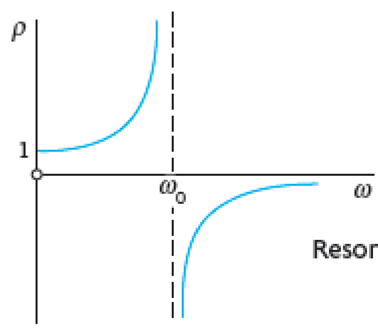
a_0 depends on ω and ω_0. If $\omega \to \omega_0$, then ρ and a_0 tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called **resonance**. ρ is called the **resonance factor** (Fig. 54), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution y_p and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.

In the case of resonance the nonhomogeneous ODE (2) becomes

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos \omega_0 t. \tag{10}$$

Then (7) is no longer valid, and, from the Modification Rule in Sec. 2.7, we conclude that a particular solution of (10) is of the form

$$y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t).$$



Resonance factor $\rho(\omega)$

By substituting this into (10) we find $a = 0$ and $b = F_0/(2m\omega_0)$. Hence (Fig. 55)

$$y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t. \tag{11}$$

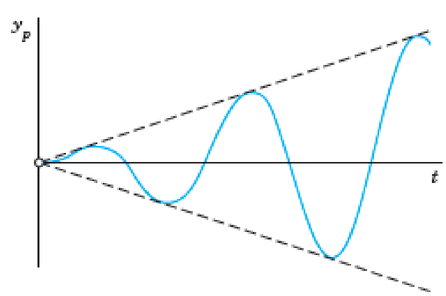


Fig. 55. Particular solution in the case of resonance

We see that, because of the factor *t*, the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system. We shall return to this practical aspect of resonance later in this section.

Beats. Another interesting and highly important type of oscillation is obtained if ω is close to ω_0. Take, for example, the particular solution [see (8)]

$$y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t) \qquad (\omega \neq \omega_0). \tag{12}$$

Using (12) in App. 3.1, we may write this as

$$y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right).$$

Since ω is close to ω_0, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 56, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they ***tune*** their instruments.

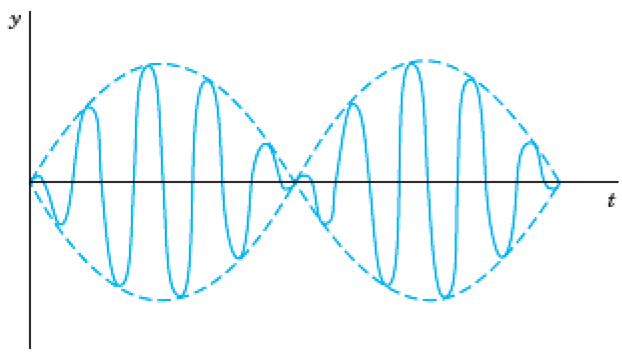


Fig. 56. Forced undamped oscillation when the difference of the input and natural frequencies is small (“**beats**”)

Case 2. Damped Forced Oscillations

If the damping of the mass–spring system is not negligibly small, we have $c > 0$ and a damping term cy' in (1) and (2). Then the general solution y_h of the homogeneous ODE (1) approaches zero as t goes to infinity, as we know from Sec. 2.4. Practically, it is zero after a sufficiently long time. Hence the “**transient solution**” (6) of (2), given by $y = y_h + y_p$, approaches the “**steady-state solution**” y_p. This proves the following.

To study the amplitude of y_p as a function of ω, we write (3) in the form

$$y_p(t) = C^* \cos(\omega t - \eta). \tag{13}$$

C^* is called the **amplitude** of y_p and η the **phase angle** or **phase lag** because it measures the lag of the output behind the input. According to (5), these quantities are

$$\begin{aligned} C^*(\omega) &= \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}, \\ \tan \eta(\omega) &= \frac{b}{a} = \frac{\omega c}{m(\omega_0^2 - \omega^2)}. \end{aligned} \tag{14}$$

Let us see whether $C^*(\omega)$ has a maximum and, if so, find its location and then its size. We denote the radicand in the second root in C^* by R. Equating the derivative of C^* to zero, we obtain

$$\frac{dC^*}{d\omega} = F_0\left(-\frac{1}{2}R^{-3/2}\right)[2m^2(\omega_0^2 - \omega^2)(-2\omega) + 2\omega c^2].$$

The expression in the brackets [. . .] is zero if

$$c^2 = 2m^2(\omega_0^2 - \omega^2) \qquad (\omega_0^2 = k/m). \tag{15}$$

By reshuffling terms we have

$$2m^2\omega^2 = 2m^2\omega_0{}^2 - c^2 = 2mk - c^2.$$

The right side of this equation becomes negative if $c^2 > 2mk$, so that then (15) has no real solution and C^* decreases monotone as ω increases, as the lowest curve in Fig. 57 shows. If c is smaller, $c^2 < 2mk$, then (15) has a real solution $\omega = \omega_{\max}$, where

$$\omega_{\max}^2 = \omega_0^2 - \frac{c^2}{2m^2}. \tag{15*}$$

From (15*) we see that this solution increases as c decreases and approaches ω_0 as c approaches zero. See also Fig. 57.

The size of $C^*(\omega_{\max})$ is obtained from (14), with $\omega^2 = \omega_{\max}^2$ given by (15*). For this ω^2 we obtain in the second radicand in (14) from (15*)

$$m^2(\omega_0^2 - \omega_{\max}^2)^2 = \frac{c^4}{4m^2} \qquad \text{and} \qquad \omega_{\max}^2 c^2 = \left(\omega_0^2 - \frac{c^2}{2m^2}\right)c^2.$$

The sum of the right sides of these two formulas is

$$(c^4 + 4m^2\omega_0^2c^2 - 2c^4)/(4m^2) = c^2(4m^2\omega_0^2 - c^2)/(4m^2).$$

Substitution into (14) gives

$$C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0^2 - c^2}}. \tag{16}$$

We see that $C^*(\omega_{\max})$ is always finite when $c > 0$. Furthermore, since the expression

$$c^2 4m^2\omega_0^2 - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as c^2 ($<2mk$) goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1. Figure 57 shows the **amplification** C^*/F_0 (ratio of the amplitudes of output and input) as a function of ω for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant c.

Figure 58 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$.

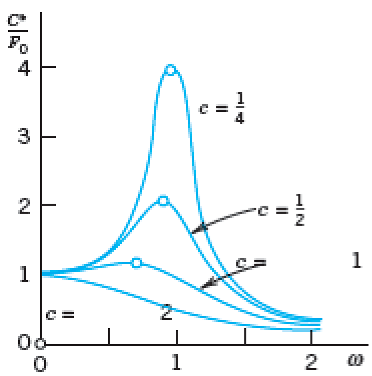


Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

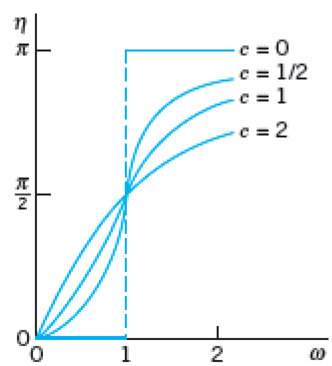


Fig. 58. Phase lag η as a function of ω for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c